DYNAMIC-THERMAL PROBLEM FOR A NON-NEWTONIAN FLUID

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The equations of the dynamic and thermal boundary layers on a flat plate are examined for a fluid with a temperature-dependent rheological power law and a non-Fourier law of heat conduction. They are reduced to ordinary differential equations and solved numerically. An asymptotic formula is obtained for calculating $\theta'(0)$. The effect of the nonisothermicity of the parameter a on the velocity and temperature profiles and on the drag and heat transfer coefficients is investigated.

In [2] Shul'man and Berkovskii examined the equations of the dynamic and thermal boundary layers on a flat plate in a fluid obeying a rheological power law and the associated non-Fourier law of heat conduction. In the present paper the same problem is investigated with a view to determining the effect of the temperature dependence of the consistency coefficient on the dynamic and thermal flow characteristics.

In relation to the boundary layer the laws of internal friction and heat conduction are written as

$$\tau = K\{\exp[-b(T-T_\infty)](\partial u_1/\partial y_1)\}^n, \tag{1}$$

$$q = -H(\partial u_1/\partial y_1)^{n-1}(\partial T/\partial y_1). \tag{2}$$

Form (1) was used in [1]; when $T = T_\infty$ it goes over into the usual power law.

Reducing the equations of the dynamic and thermal boundary layers to dimensionless form using (1) and (2), we obtain

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[\exp(-a\theta)\frac{\partial u}{\partial y}\right]^n, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3}$$

$$u\frac{\partial\theta}{\partial x} + v\frac{\partial\theta}{\partial y} = \frac{1}{\sigma}\frac{\partial}{\partial y}\left[\left(\frac{\partial u}{\partial y}\right)^{n-1}\frac{\partial\theta}{\partial y}\right]. \tag{4}$$

Here

$$u = \frac{u_1}{U}, \quad v = \frac{v_1}{U}R^{\frac{1}{1+n}},$$

$$x = \frac{x_1}{L}, \quad y = \frac{y_1}{L}R^{\frac{1}{1+n}}, \quad \sigma = \frac{Kc_p}{H},$$

$$a = b(T_w - T_\infty), \quad \theta = \frac{T - T_\infty}{T_w - T_\infty}, \quad R = \frac{\rho U^{2-n}L^n}{K}. \tag{5}$$

The boundary conditions of Eqs. (3), (4) are as follows:

$$u = v = 0, \ \theta = 1 \ \text{ at } \ y = 0,$$
$$u = 1, \ \theta = 0 \ \text{ at } \ y = \infty. \tag{6}$$

System (3), (4) admits a similar solution

$$u = \varphi'(\eta), \ \theta = \theta(\eta), \ \ \eta = y[n(1+n)x]^{-\frac{1}{1+n}} \tag{7}$$

(a prime denotes the derivative with respect to η).

From the second of Eqs. (3), using (6) and (7), we find that

$$v = [n(1+n)x]^{\frac{1}{1+n}}[(1+n)x]^{-1}[\eta\varphi' - \varphi]. \tag{8}$$

With Eqs. (7) and (8) we reduce (3), (4), and (6) to a system of coupled ordinary differential equations

$$\varphi''' = [a\theta' - \exp(a_n\theta)\varphi(\varphi'')^{1-n}]\varphi'', \tag{9}$$

$$\theta'' = [(1-n)\varphi'''/\varphi'' - \sigma_n\varphi(\varphi'')^{1-n}]\theta' \tag{10}$$

with boundary conditions

$$\varphi = \varphi' = 0, \ \theta = 1 \ \text{ at } \ \eta = 0, \tag{11}$$

$$\varphi' = 1, \ \theta = 0 \ \text{ at } \ \eta = \infty. \tag{12}$$

when $a = 0$ Eq. (9) does not depend on (10) and goes over into the generalized Blasius equation [2].

In practice, system (9), (10) has been solved for condition (11) and condition

$$\varphi' \to 1 \ \text{ as } \ \varphi'' \to 0, \ \theta \to 0 \ \text{ as } \ \theta' \to 0, \tag{13}$$

Table 1

Values of $\varphi''(0)$ (Upper) and $\theta'(0)$ (Lower)

n	σ = 10			30			100		
	a = 0	0.5	1.0	0	0.5	1.0	0	0.5	1.0
0.6	0.3157 0.6563	0.4847 0.8082	0.7339 0.9899	0.3157 0.9441	0.4992 1.1784	0.7814 1.4653	0.3157 1.4091	0.5094 1.7753	0.8169 2.2321
0.8	0.3962 0.8511	0.6132 0.9954	0.9331 1.1581	0.3962 1.2274	0.6297 1.4507	0.9890 1.7093	0.3962 1.8336	0.6410 2.1825	1.0299 2.5949
1.0	0.4696 1.0297	0.7311 1.1421	1.1150 1.2619	0.4696 1.4873	0.7490 1.6633	1.1782 1.8572	0.4696 2.2229	0.7612 2.4996	1.2242 2.8119
1.5	0.6189 1.3958	0.9721 1.3523	1.4841 1.3159	0.6189 2.0200	0.9925 1.9658	1.5615 1.9256	0.6189 3.0209	1.0061 2.9478	1.6181 2.8996

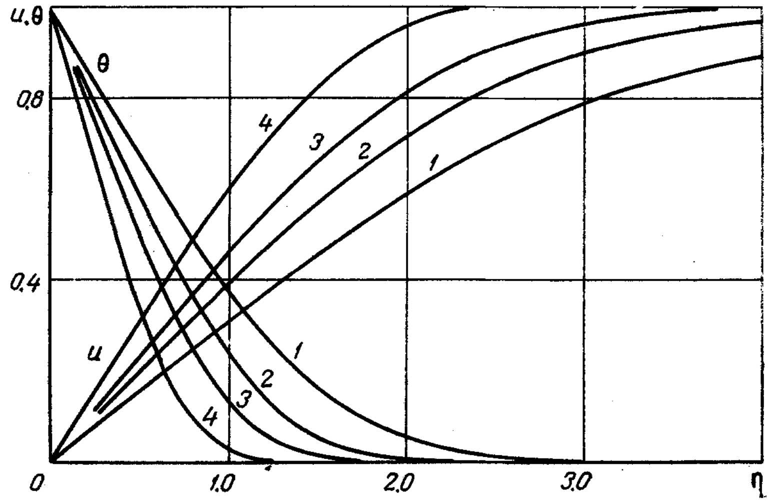

Fig. 1. Velocity and temperature profiles in the boundary layer at $a = 0$, $\sigma = 10$: 1) n = 0.6; 2) 0.8; 3) 1.0; 4) 1.5.

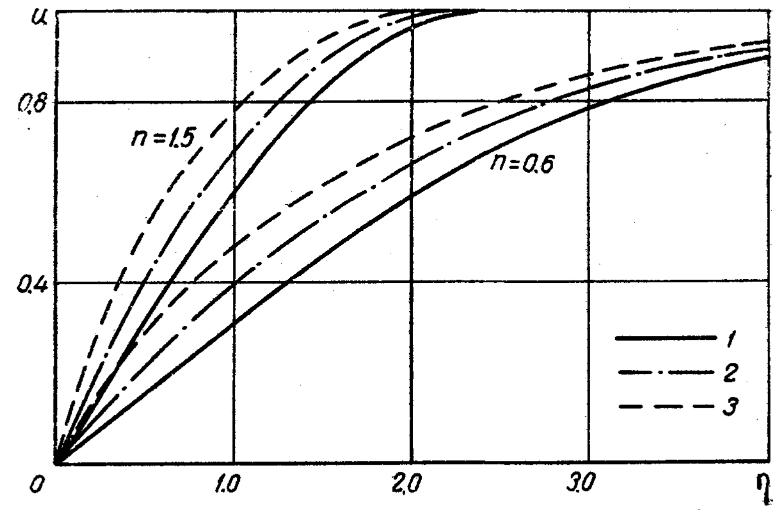

Fig. 2. Effect of a on the velocity profiles at $\sigma = 10$: 1) $a = 0$; 2) 0.5; 3) $a = 1$.

Table 2

Comparison of Exact and Approximate Values of $\theta'(0)$

n	σ = 10 exact	10 equation (17)	10 error %	30 exact	30 equation (17)	30 error %	100 exact	100 equation (17)	100 error %
0.6	0.6563	0.6537	—0.43	0.9441	0.9428	—0.14	1.409	1.408	—0.07
0.8	0.8511	0.8508	—0.04	1.227	1.227	—	1.834	1.833	—0.05
1.0	1.030	1.032	+0.19	1.487	1.488	+0.07	2.223	2.223	—
1.5	1.396	1.403	+0.50	2.020	2.023	+0.15	3.021	3.022	+0.03

which is applicable both at $n \le 1$ and at $n > 1$, when the thicknesses of the dynamical and thermal boundary layers are finite. The solution was obtained on a "Ural-2" computer using the Runge-Kutta method, the unknowns $\varphi''(0)$ and $\theta'(0)$ being found by Newton's method. The results are shown in Table 1.

Figures 1–4 present examples of u and θ profiles characterizing the effect of the parameters n, a, and σ. Figure 1 gives the ordinary distributions of u and θ for various n. Figures 2 and 3 characterize the effect of a on u and θ. We see that a affects the θ profiles less than the u profiles, especially at $n > 1$. In accordance with (9) and (10), when $a \neq 0$ the u and θ profiles begin at $\eta = 0$ with nonzero curvature. The parameter a does not affect the boundary layer thicknesses owing to the selected form of law (1). Figure 4 characterizes the effect of σ on θ and u at $a \neq 0$. As

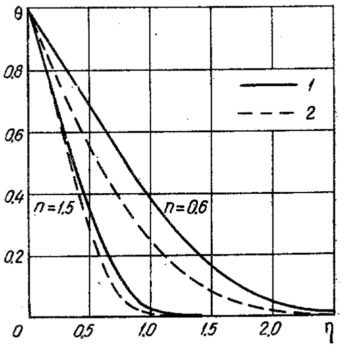

Fig. 3. Effect of a on the temperature profiles at $\sigma = 10$: 1) $a = 0$; 2) $a = 1$.

usual, an increase in σ leads to a decrease in the thickness of the thermal boundary layer; naturally, σ does not affect the thickness of the dynamic boundary layer, but has some influence on the u profile.

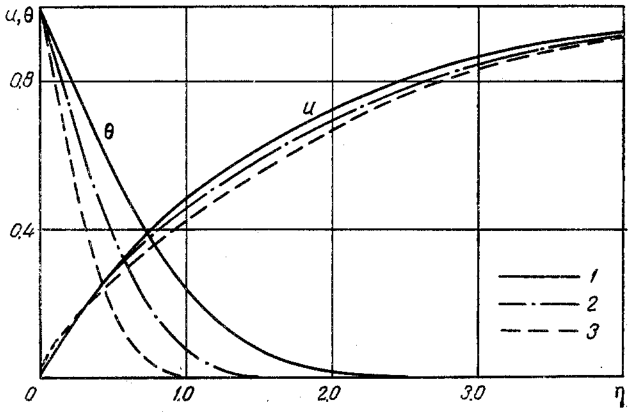

Fig. 4. Effect of σ on the velocity and the temperature profiles at $n = 0.6$; $a = 1$: 1) $\sigma = 10$; 2) 30; 3) 100.

In the range of σ investigated at $a = 0$ the θ profiles lie in the almost linear region of the u profiles. This property makes it possible to solve Eq. (10) approximately as follows.

We set $u' = \varphi'' = \text{const} = \beta$. Then, using (11)

$$\varphi''' = 0,\ \varphi' = \beta\eta,\ \varphi = \beta\eta^2/2. \qquad (14)$$

Substituting (14) into (10), after integration, using (11), we obtain

$$\theta = C\int_0^\eta \exp(-k\eta^3)\,d\eta + 1,\ k = \sigma_n\beta^{2-n}/6. \qquad (15)$$

The constant $C = \theta'(0)$ is found from condition (12). The substitution $k\eta^3 = t$ transforms (15) to

$$\theta = \frac{C}{3k^{1/3}}\int_0^t t^{1/3-1}\exp(-t)\,dt + 1, \qquad (16)$$

Table 3

Values of B (Upper) and E (lower)

n	σ = 10, a = 0	10, 0.05	10, 1.0	30, 0	30, 0.5	30, 1.0	100, 0	100, 0.5	100, 1.0
0.6	1.017	0.9740	0.9257	1.017	0.9915	0.9612	1.017	1.004	0.9874
	1.057	1.096	1.137	1.520	1.580	1.642	2.269	2.361	2.458
0.8	0.8108	0.7710	0.7230	0.8108	0.7875	0.7574	0.8108	0.7988	0.7824
	0.8710	0.9333	0.9986	1.256	1.353	1.457	1.876	2.028	2.194
1.0	0.6641	0.6270	0.5801	0.6641	0.6425	0.6130	0.6641	0.6530	0.6369
	0.7282	0.8076	0.8925	1.052	1.176	1.313	1.572	1.768	1.988
1.5	0.4405	0.4096	0.3650	0.4405	0.4226	0.3939	0.4405	0.4313	0.4155
	0.4968	0.6032	0.7252	0.7189	0.8861	1.089	1.075	1.338	1.669

whence using (12) we have

$$-\theta'(0) = 3/\Gamma(1/3)(\sigma n \beta^{2-n}/6)^{1/3} \cong$$
$$\cong 0.6161(\sigma n \beta^{2-n})^{1/3}. \qquad (17)$$

Equation (17) is an asymptotic formula for $\sigma \gg 1$ analogous to Lighthill's formula [3]. As may be seen from Table 2, (17) is fairly accurate, depends on n, and increases significantly with increase in σ.

Using (1), (2), (5), and (7) we find the local friction drag and heat-transfer coefficients for the plate:

$$c_f = 2\tau_w/\rho U^2 = BR_{x_1}^{-\frac{1}{1+n}}, \qquad (18)$$

$$\mathrm{Nu/R} = -q_w K/H\rho U(T_w - T_\infty) = ER_{x_1}^{-\frac{1}{1+n}}, \qquad (19)$$

where

$$B = 2[n(1+n)]^{-\frac{n}{1+n}} \exp(-an)[\varphi''(0)]^n, \qquad (20)$$

$$E = -[n(1+n)]^{-\frac{n}{1+n}} [\varphi''(0)]^{n-1}\theta'(0). \qquad (21)$$

Values of B and E are presented in Table 3. We see that owing to the presence of the factor exp($-an$) associated with $[\varphi''(0)]^n$ in (20) and the factor $[\varphi''(0)]^{n-1}$ associated with $\theta'(0)$ in (21) a has much less effect on B and E than on $\varphi''(0)$ and $\theta'(0)$ (Table 1).

NOTATION

τ and q are the friction and heat flux in the boundary layer, respectively; τ_w and q_w are the same at the wall; K, n are the rheological characteristics of the fluid; b is a constant; H is the heat conduction characteristic; x_1 is the longitudinal coordinate; y_1 is the transverse coordinate; u_1, v_1 are the velocity vector components along the x_1 and y_1 axes, respectively; U is the free-stream velocity; L is the characteristic length; R is the Reynolds number; R_{x_1} is the local Reynolds number; T is the absolute temperature; T_w is the same at the wall; T_∞ is the same in the free stream; and Nu is the Nusselt number.

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